Monday April 14, exam I, MAT 335

100

100

① (35%) Consider the IBVP

PDE: $u_t = u_{xx} - u + x \qquad 0 < x < 1, \qquad 0 < t < \infty$

BCs: $u(0,t) = 0 \qquad 0 < t < \infty$

$u(1,t) = 1 \qquad 0 < t < \infty$

IC: $u(x,0) = 0 \qquad 0 \le x \le 1$

a) Let $u(x,t) = x + e^{-t} \cdot w(x,t)$. Reformulate the above IBVP in terms of w.

b) Use the method of separation of variables to find w. Show all details.

a) $u(x,t) = x + e^{-t} w(x,t)$

$-e^{-t} w + e^{-t} w_t = e^{-t} w_{xx} - x - e^{-t} w + x$

$\therefore \quad w_t = w_{xx}$

$u(0,t) = 0 + e^{-t} w(0,t) = 0 \quad w(0,t) = 0$

$u(1,t) = 1 + e^{-t} w(1,t) = 1 \quad w(1,t) = 0$

$u(x,0) = x + w(x,0) = 0$

$w(x,0) = -x$

$$\begin{cases} w_t = w_{xx} \\ w(0,t) = 0 \\ w(1,t) = 0 \\ w(x,0) = -x \end{cases}$$

b) $w(x,t) = X(x)T(t)$

$XT' = TX''$

$\frac{T'}{T} = \frac{X''}{X} = K$ (because x and t are independent)

$T' - KT = 0 \quad T = Ce^{Kt}$ K should be negative or else T and u will blow up when $t \to \infty$

$X'' - KX = 0 \quad T = Ce^{-\lambda^2 t}$

$$w(x,t) = e^{-\lambda^2 t}\left[A\sin\lambda x + B\cos\lambda x\right]$$

$$w(0,t) = 0 \therefore B e^{-\lambda^2 t} = 0$$
$$B = 0$$

$$w(x,t) = Ae^{-\lambda^2 t}\sin\lambda x$$

$$w(1,t) = 0 \therefore Ae^{-\lambda^2 t}\sin\lambda = 0$$
$$\sin\lambda = 0$$
$$\lambda = \lambda_n = n\pi$$

$$w(x,t) = A_n e^{-(n\pi)^2 t}\sin n\pi x$$

IC $w(x,0) = A_n \sin n\pi x = -x$
$$A_n = f(x) \longrightarrow \longleftarrow$$

By superposition principle, as PDE and BCs are linear and homogeneous $\therefore$ $w(x,t) = \sum A_n e^{-(n\pi)^2 t}\sin n\pi x \quad n \geq 1$

IC $w(x,0) = \sum A_n \sin n\pi x = -x$

To get A_n we multiply by $\sin n\pi x$, integrate from 0 to 1 and we use the orthogonality property of sin.

$$A_n = 2\int_0^1 -x \sin n\pi x \, dx$$

$$= -2\int_0^1 x \sin n\pi x \, dx$$

x	$\sin n\pi x$
1	$\frac{-1}{n\pi}\cos n\pi x$
0	$\frac{-1}{(n\pi)^2}\sin n\pi x$

$$= -2\left[\frac{-x}{n\pi}\cos n\pi x + \frac{1}{(n\pi)^2}\sin n\pi x\right]_0^1$$

$$= -2\left[\frac{-1}{(n\pi)}(-1)^n\right] = \frac{2(-1)^{n+2}}{n\pi} = \begin{cases} 2/n\pi & n \text{ even} \\ -2/n\pi & n \text{ odd} \end{cases}$$

$$w(x,t) = \sum A_n e^{-(n\pi)^2 t}\sin n\pi x \quad \text{with } A_n = \begin{cases} 2/n\pi & n \text{ even} \\ -2/n\pi & n \text{ odd} \end{cases}$$

(2) (35%)

a) Use the Fourier transform to solve the IVP

PDE: $u_t = \alpha^2 u_{xx} - \beta u \qquad -\infty < x < \infty, \quad 0 < t < \infty$

IC: $u(x,0) = \phi(x) \qquad -\infty < x < \infty$

Show all details.

b) What is the solution in the special case $\phi(x) \equiv 1$?

a) $u_t = \alpha^2 u_{xx} - \beta u$.

Let $U(\xi,t) = \mathcal{F}(u(x,t))(\xi) \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)\, e^{-i\xi x}\, dx$.

$\mathcal{F}(PDE) \Rightarrow U_t(\xi,t) = -\alpha^2\xi^2 U(\xi,t) - \beta U(\xi,t)$

$U_t(\xi,t) + (\alpha^2\xi^2 + \beta)\, U(\xi,t) = 0$.

1st order linear homogeneous PDE with constant coefficients

$U(\xi,t) = c\, e^{-(\alpha^2\xi^2+\beta)t}$

IC: $u(x,0) = \phi(x) \quad U(\xi,0) = \Phi(\xi) \quad \Phi(\xi) = \mathcal{F}(\phi(x))$

$\therefore U(\xi,0) = c = \Phi(\xi)$

$\therefore U(\xi,t) = \Phi(\xi)\, e^{-(\alpha^2\xi^2+\beta)t}$

$u(x,t) = \mathcal{F}^{-1}(U(\xi,t)) = \mathcal{F}^{-1}(\Phi(\xi)) * \mathcal{F}^{-1}\left(e^{-(\alpha^2\xi^2+1)t}\right)$

$= \phi(x) * \mathcal{F}^{-1}\left(e^{-(\alpha^2\xi^2+\beta)t}\right)$.

$= \Phi(x) * \mathcal{F}^{-1}\left(e^{-\alpha^2 t\xi^2} \cdot e^{-\beta t}\right)$

$= \Phi(x) * e^{-\beta t}\, \mathcal{F}^{-1}\left(e^{-\alpha^2 t \xi^2}\right)$

~~$u(x,t) = \Phi(x)*$~~ $e^{-\beta t}$

$\mathcal{F}^{-1}\left(e^{-\frac{\xi^2}{4a^2}}\right) = a\sqrt{2}\, e^{-a^2x^2}$

(3) (30%) Use the Laplace transform to solve the IVBP

PDE: $u_{xx} - u_{tt} = -\sin \pi x \qquad 0 < x < 1, \quad 0 < t < \infty$

BCs: $u(0,t) = u(1,t) = 0 \qquad 0 < t < \infty$

ICs: $u(x,0) = u_t(x,0) = 0 \qquad 0 \le x \le 1, \quad 0 < t < \infty$

Let $\mathcal{U}(x,s) = \mathcal{L}(u(x,t))(s) = \int_0^{\infty} u(x,t) e^{-st}\,dt$ $\quad \mathcal{L}(-\sin\pi x) = -\sin \pi x\,\mathcal{L}(1) = -\frac{\sin\pi x}{s}$

$\mathcal{L}(PDE) \Rightarrow \mathcal{U}_{xx}(x,s) - \left[s^2 \mathcal{U}(x,s) - s u(x,0) - u_t(x,0)\right] = -\frac{\sin\pi x}{s}$

$\mathcal{U}_{xx}(x,s) - s^2 \mathcal{U}(x,s) + 0 = -\frac{\sin \pi x}{s}$

$\mathcal{U}_{xx}(x,s) - s^2 \mathcal{U}(x,s) = -\frac{\sin\pi x}{s}$

2nd order linear nonhomogeneous ODE with variable coefficients

$\mathcal{U}(x,s) = A e^{-sx} + B e^{sx} + c_1(x) e^{-sx} + c_2(x) e^{sx}$

$c_1(x) = \frac{1}{2s}\int -\frac{\sin\pi x}{s} e^{sx}\,dx = \frac{1}{2s^2}\int \sin \pi x\, e^{sx}\,dx$

$= \frac{1}{2s^2}\left[\frac{e^{sx}}{s^2+\pi^2}\left(s\sin\pi x - \pi\cos\pi x\right)\right]$

$c_2(x) = \frac{1}{2s}\int -\frac{\sin\pi x}{s} e^{-sx}\,dx = \frac{-1}{2s^2}\left[\frac{e^{-sx}}{s^2+\pi^2}\left[-s\sin\pi x - \pi\cos\pi x\right]\right]$

$= \frac{e^{-sx}}{2s^2(s^2+\pi^2)}\left[s\sin\pi x + \pi\cos\pi x\right]$

$\mathcal{U}(x,s) = A e^{-sx} + B e^{sx} + \frac{1}{2s^2(s^2+\pi^2)}(s\sin\pi x - \pi\cos\pi x) + \frac{1}{2s^2(s^2+\pi^2)}(s\sin\pi x + \pi\cos\pi x)$

$\mathcal{U}(x,s) = A e^{-sx} + B e^{sx} - \frac{s \sin\pi x}{[illegible]} = A e^{-sx} + B e^{sx} - \frac{\sin \pi x}{[illegible]}$

5/6

~~$\mathcal{U}(x,s) =$~~ $\mathcal{U}(0,s) = 0$

$u(0,t)=0 \Rightarrow$ ~~$A+B+\sin\pi$~~ $\Rightarrow A+B+\frac{\cancel{\sin\pi x}^{\,0}}{s(s^2+\pi^2)} = 0$

$\boxed{A=-B}$

$u(1,t)=0 \Rightarrow \mathcal{U}(1,s)=0 \Rightarrow Ae^{-s}+Be^{s}+\frac{\cancel{\sin\pi}^{\,0}}{s(s^2+\pi^2)} = 0$

$Ae^{-s} = -Be^{s}$

~~$Be^{-s} = Be^{s}$~~

$A = -Be^{2s}$

$A=-B \quad \boxed{-B = -Be^{2s}} \Rightarrow e^{2s}=1$ $s=0 \rightarrow$ c

For this to be satisfied for any s $\boxed{B=0}$

$\boxed{A=0}$

$\mathcal{U}(x,s) = \frac{\sin\pi x}{s(s^2+\pi^2)}$

$u(x,t) = \mathcal{L}^{-1}\left(\frac{\sin\pi x}{s(s^2+\pi^2)}\right) = \sin\pi x \; \mathcal{L}^{-1}\left(\frac{1}{s(s^2+\pi^2)}\right)$

$= \sin\pi x \; \mathcal{L}^{-1}\left(\frac{1}{s}\,\frac{1}{s^2+\pi^2}\right)$ multiply up and down by π

$= \sin\pi x \; \mathcal{L}^{-1}\left(\frac{1}{\pi s}\,\frac{\pi}{s^2+\pi^2}\right)$

$= \sin\pi x \; \mathcal{L}^{-1}\left(\frac{1}{\pi s}\right) * \mathcal{L}^{-1}\left(\frac{\pi}{s^2+\pi^2}\right)$

$= \sin\pi x \left(\frac{1}{\pi} * \sin\pi t\right)$ $\quad f(t-\tau) = \frac{1}{\pi}$

$= \sin\pi x \int_0^t \frac{1}{\pi}\sin(\pi\tau)\,d\tau$

$u(x,t) = \sin\pi x \; \frac{-1}{\pi^2}\cos\pi\tau\Big|_0^t$

$u(x,t) = \frac{-\sin\pi x}{\pi^2}(\cos\pi t - 1) = \frac{\sin\pi x}{\pi^2}(1-\cos$